# ON TORSION IN THE INTERSECTION COHOMOLOGY OF SCHUBERT VARIETIES 

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#### Abstract

We prove that the prime torsion in the local integral intersection cohomology of Schubert varieties in the flag variety of the general linear group grows exponentially in the rank. The idea of the proof is to find a highly singular point in a Schubert variety and calculate the Euler class of the normal bundle to the (miraculously smooth) fibre in a particular Bott-Samelson resolution. The result is a geometric version of an earlier result established using Soergel bimodule techniques.


Dedicated to the memory of Sandy Green.

## 1. Introduction

Let $X$ be a projective complex algebraic variety equipped with its metric topology and let $H^{*}(X, \mathbb{Q})$ denote its rational cohomology ring. If $X$ is smooth then there are several remarkable and useful theorems concerning $H^{*}(X, \mathbb{Q})$ : Poincaré duality, the hard Lefschetz theorem, the Hodge decomposition, the Hodge-Riemann relations. If instead one takes integral coefficients then (derived) Poincaré duality still holds.

None of these theorems are valid for singular $X$. Instead one can consider the intersection cohomology $I H^{*}(X, \mathbb{Q})$ of $X$ as defined by Goresky and MacPherson. If $X$ is smooth then one has a canonical identification between cohomology and intersection cohomology. Goresky and MacPherson proved that Poincaré duality always holds in rational intersection cohomology. It was subsequently discovered that (analogues of the) the hard Lefschetz theorem, the Hodge decomposition and the Hodge-Riemann relations all hold in intersection cohomology [BBD82, Sai89]. Thus it is not surprising that intersection cohomology provides a powerful complement to ordinary cohomology in the study of singular algebraic varieties.

As with ordinary cohomology, it is also possible to define intersection cohomology groups with coefficients in any field or the integers. In their original paper on intersection (co)homology, Goresky and MacPherson noticed that (derived) Poincaré duality does not hold over the integers for intersection cohomology [GM80, 6.3]. For example, if $X$ is smooth then Poincaré duality implies that the intersection form on (the free part of) its middle cohomology is unimodular (i.e. non-degenerate over $\mathbb{Z}$ ). Goresky and MacPherson gave an example to show that the analoguous statement need not hold for integral intersection cohomology. For a given singular $X$ it is appears to be a difficult question to decide whether its integral intersection cohomology satisfies Poincaré duality over the integers, or for which primes $p$ it fails.

This question has a local variant. Just as the ordinary cohomology of a space can be described as the cohomology of the constant sheaf, intersection cohomology may be obtained as the hypercohomology of the intersection cohomology complex,
a constructible complex of abelian groups on $X$. Let $\mathbf{I C}(X, \mathbb{Z})$ denote the integral intersection cohomology complex of $X$. The local variant of the above question which we consider in this paper is the following:

Question 1.1. Descibe the $p$-torsion in the stalks or costalks of $\mathbf{I C}(X, \mathbb{Z})$. In particular, for which primes $p$ are all the stalks and costalks free of $p$-torsion?

Some remarks about this question are in order:
(1) If there is no torsion in the stalks or costalks of $\mathbf{I C}(X, \mathbb{Z})$ then the integral intersection cohomology satisfies Poincaré duality. The $p$-local version of this statement also holds: absence of $p$-torsion implies that Poincaré duality holds after inverting all primes $\neq p$. These statements are not if and only if in general, however in the case of Schubert varieties (considered below) they are.
(2) In general the rational intersection cohomology complex $\mathbf{I C}(X, \mathbb{Q})$ is much easier to describe. This is due to the decomposition theorem [BBD82], which allows one to compute $\mathbf{I C}(X, \mathbb{Q})$ via resolutions. In particular in many cases $\mathbf{I C}(X, \mathbb{Q})$ can be considered "known" and the above question asks whether $\mathbf{I C}(X, \mathbb{Z})$ contains any surprises.
(3) For general $X$ this question appears to be very hard. For example, if $X$ has only isolated singularities then the question is equivalent to understanding the torsion in the cohomology of the links ${ }^{1}$ to all singular points. In general the link of an isolated singularity can be a rather complicated manifold, and describing the torsion in its cohomology can be a difficult task.
As well as their intrinsic interest, these questions have applications in the modular representation theory of finite and algebraic groups. Starting with the KazhdanLusztig conjectures, intersection cohomology methods have been very fruitful in Lie theory (see [Lus91] for an impressive list of applications). The power of these methods in characteristic zero representation theory is usually thanks to the decomposition theorem. More recently it has been suggested that similar methods could be used to attack questions in modular representation theory [Soe00, MV07, Jut14, JMW12]. However here the decomposition theorem is missing. The above questions asks for obstructions to transporting out knowledge in characteristic zero to knowledge in characteristic $p$.

In geometric representation theory a central role is played by Schubert varieties. The goal of this paper is to provide the following partial answer to the above question in this case:

Theorem 1.2. The p-torsion (for $p$ a prime) in the stalks and costalks of the integral intersection cohomology complexes on Schubert varieties in the flag variety of the general linear group grows at least exponentially in the rank.

Again, some remarks are in order:
(1) This is a geometric version of an earlier theorem proved using Soergel bimodules, diagrammatics and the nil Hecke ring in [HW15, Wil13]. Important contributions to this theorem were made by Soergel, Libedinsky,

[^0]Elias-Khovanov, Elias and He. On the geometric side important contributions were made by Braden (who discovered 2-torsion for $n=8$, see the appendix to [WB12]) and Polo (who showed the existence of $n$-torsion in rank $4 n$ ).
(2) Schubert varieties admit affine pavings and so their ordinary cohomology is free over the integers. The above theorem tells us that (at least locally) there is lots of torsion in intersection cohomology.
(3) We are still very far from a complete understanding of Question 1.1 in the setting of Schubert varieties. This is already evident in the phrase "at least" in the above theorem, whose proof produces many examples of torsion, but certainly makes no claim to exhaustiveness.
(4) In many "simple" examples (Schubert varieties in Grassmannians [Zel83], Schubert varieties for $G L(n, \mathbb{C})$ for $n \leq 7$ [WB12]) there is no torsion at all in the stalks or costalks of integral intersection cohomology sheaves. The above theorem tells us that these simple examples are rather deceptive.
(5) By results of Soergel one can use the above theorem to deduce that any bound for Lusztig's conjecture on the characters of simple rational representations of $G L_{n}$ in characteristic $p$ must grow at least exponentially in $n$. Hence the above theorem gives many counterexamples to the expected bounds in Lusztig's conjecture [Lus80]. Similarly, in [Wil13] it is explained how one can use such results to produce counterexamples to a conjecture of James [Jam90] on the simple modular representations of the symmetric group.
(6) All Schubert varieties in the flag variety of $G L_{n}$ also occur as Schubert varieties in the flag varieties of groups of types $B_{n}, C_{n}$ and $D_{n}$. Hence the above theorem may be rephrased as saying that the torsion in the stalks and costalks of the integral intersection cohomology of Schubert varieties in the flag variety of any simple complex algebraic group grows at least exponentially in the rank.

As already mentioned, the above theorem can be deduced from previous work in the context of Soergel bimodules. However I think it is worthwhile to publish a new proof for two reasons:
(1) The proof relies only on the combinatorics of expressions and geometric ideas. In particular it does not use the theory of Soergel bimodules, their diagrammatics, or the nil Hecke ring (as in [HW15, Wil13]). Hence this paper is potentially accessible to a wider audience than [Wil13].
(2) The proof provides a recipe to find many highly singular points in Schubert varieties, whose resolutions are nonetheless amenable to explicit analysis. It is possible that these points and their resolutions will be useful in other problems in singularity theory and the study of Schubert varieties. With such potential future applications in mind, and also with the goal of understanding existing work more conceptually, an explicit description of the geometry of the situation seems worthwhile.
1.1. Main theorem. We now give a more precise formulation of the main theorem of this paper. The formulation is somewhat technical, and hence we need some more notation.

Let $R=\mathbb{Z}\left[\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{n}\right]$ be a polynomial ring in $n$ variables graded such that $\operatorname{deg} \varepsilon_{i}=2$ and let $W=S_{n}$ the symmetric group on $n$-letters. Then $W$ acts by permutation of variables on $R$. Let $s_{1}, \ldots, s_{n-1}$ denote the simple transpositions of $S_{n}$ and let $\ell$ denote the length function. Let $\partial_{i}$ denote the $i^{t h}$ divided difference operator:

$$
\partial_{i}(f)=\frac{f-s_{i} f}{\varepsilon_{i}-\varepsilon_{i+1}} \in R
$$

For any element $w \in S_{n}$ we have well-defined operators $\partial_{w}=\partial_{i_{1}} \ldots \partial_{i_{m}}$ where $w=s_{i_{1}} \ldots s_{i_{m}}$ is a reduced expression for $w$.

Consider elements of the form

$$
\begin{equation*}
C=\partial_{w_{m}}\left(\varepsilon_{n}^{a_{m}} \partial_{w_{m-1}}\left(\varepsilon_{n}^{a_{m-1}} \ldots \partial_{w_{1}}\left(\varepsilon_{n}^{a_{1}}\right) \ldots\right)\right) \tag{1.1}
\end{equation*}
$$

where $w_{i} \in S_{n}$ are arbitrary. We assume that $\sum \ell\left(w_{i}\right)=a$ where $a=\sum a_{i}$. Because $\varepsilon_{i}$ has degree 2 and $\partial_{w}$ has degree $-2 \ell(w)$ it follows that $C \in \mathbb{Z}$ for degree reasons.

Let $N:=n+a$, and $G=G L_{N}(\mathbb{C})$. We identify $S_{N}$ (the symmetric group on $N$ letters) with the Weyl group of $G$ in the standard way. Let $B \subset G$ denote the Borel subgroup of upper-triangular matrices. Given any subset $I \subset\{1, \ldots, N-1\}$ we let $w_{I}$ denote the longest element of the standard parabolic subgroup of $S_{N}$ corresponding to $I$, and by $P_{I} \supset B$ denote the parabolic subgroup corresponding to $I$. Let $M:=\{1, \ldots, n-1\}$.

The main result of this paper is the following:
Theorem 1.3. Suppose that $C \neq 0$. Then there exists a Schubert variety $X \subset$ $G L_{n+a} / P_{M}$ and a (Bott-Samelson) resolution

$$
f: \widetilde{X} \rightarrow X \subset G / P_{M}
$$

such that the complex $R f_{*}\left(\mathbb{Z} / C \mathbb{Z}_{\tilde{X}}\right)$ (the derived pushforward of the constant sheaf on $\widetilde{X}$ ) is not isomorphic to a direct sum of intersection cohomology sheaves. More precisely, the decomposition theorem fails at the point $w_{I}$, where $I=\{1,2, \ldots, n-$ $1, n+1, \ldots, n+a-1\}$.

Remarks:
(1) The Schubert variety $X$ and resolution $\tilde{X}$ are explicit starting from the expression (1.1) for $C$. We refer the reader $\S 7$ for the description of all spaces involved.
(2) The failure of the decomposition theorem in Theorem 1.3 implies the existence of some intersection cohomology complex supported on $X$ which has $p$-torsion in its stalk or costalk, for some prime $p$ dividing $C$. This fact can be easily deduced from the theory of parity sheaves [JMW14].
(3) Given the above theorem it is easy to produce many examples of $C$ with large prime factors relative to $N=n+a$ (see [Wil13, §6]). For example, one can find expressions which produce Fibonacci numbers linearly in $N$. However to prove that these factors grow exponentially with respect to $N$ requires some rather sophisticated results from number theory. This is discussed in detail in the appendix to [Wil13] by Kontorovich, McNamara and the author.
(4) A technical point: The allowed expressions for $C$ in [Wil13] are slightly more general than those allowed above; in [Wil13] one is also allowed expressions involving both $\varepsilon_{n}$ and $\varepsilon_{1}$, and not only $\varepsilon_{n}$ as above. However I have checked
that there are no essential gains by allowing these more general expressions, and the setting of the current paper simplifies proofs.
1.2. Acknowledgements: Expressions of the form (1.1) emerged first in joint work with Xuhua He [HW15]. Subsequently I tried to find a geometric explanation, which is the Euler class lemma of $\S 8$. I would like to thank him for many useful discussions and observations. I am also grateful to Tom Braden, Daniel Juteau, Carl Mautner and Patrick Polo from whom I learnt most of the geometric and topological techniques of this paper.

It is a great pleasure to dedicate this paper to the memory of Sandy Green. One of my first memories of representation theory is Gus Lehrer's empassioned description of Green functions and the character table of the finite general linear group. I spent 2008-2011 as a postdoc in Oxford and I remember Sandy's active participation in the representation theory seminar. After one of my first lectures on parity sheaves he excitedly asked many questions, and expressed his desire to better understand perverse sheaves. I was impressed at his openness to new ideas, and have tried to imitate it since.

## 2. Notation

Varieties and sheaves: Throughout all algebraic varieties are over $\mathbb{C}$ and are equipped with their classical (metric) topologies. Dimension and codimension always refer to complex dimension. Given a ring $\Lambda$ and a space $X$ we denote by $\Lambda_{X}$ the constant sheaf on $X$ with values in $\Lambda$.

Expressions and subexpressions: Throughout we view the symmetric group $S_{n}=$ $W$ as a Coxeter group with simple reflections $S \subset W$ the simple transpositions. An expression is a sequences $\underline{w}=\left(s_{1}, \ldots, s_{m}\right)$ with $s_{i} \in S$. We write expressions as $\underline{w}=s_{1} \ldots s_{m}$ and dropping the underline denotes the product $w \in W$. Given a fixed subexpression $\underline{w}=s_{1} \ldots s_{m}$ a subexpression is a sequence $\underline{e}=e_{1} \ldots e_{m}$ with each $e_{i} \in\{0,1\}$. What is traditionally referred to as a subexpression is the sequence $\left(s_{1}^{e_{1}}, \ldots, s_{m}^{e_{m}}\right)$, however we prefer the more economical notation. We write $\underline{e} \subset \underline{w}$ to indicate that $\underline{e}$ is a subexpression of $\underline{w}$. Given $\underline{e}=e_{1} \ldots e_{m} \subset \underline{w}$ we set $\underline{w}^{\underline{e}}=s_{1}^{e_{1}} \ldots s_{m}^{e_{m}}$.

## 3. What we need to do

In this section we recall some standard material on the role played by intersection forms in the decomposition theorem. This section gives the algebro-geometric scaffolding of the rest of the paper. One can find background material for this section in [BBD82, dCM02, CG97, JMW14].

Fix a (singular) normal and irreducible complex algebraic variety $X$ and a resolution of singularities ${ }^{2}$

$$
f: \widetilde{X} \rightarrow X
$$

We fix a stratification of $X$ adapted to $f$, i.e. a stratification

$$
X=\bigsqcup X_{\lambda}
$$

[^1]of $X$ into a finite disjoint union of locally closed, connected and smooth subvarieties such that the induced map $f: f^{-1}\left(X_{\lambda}\right) \rightarrow X_{\lambda}$ is a topologically locally trivial fibration in (usually singular) varieties.

By the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber, $R f_{*} \mathbb{Q}_{\tilde{X}}$ is isomorphic to a direct sum of shifts of intersection cohomology sheaves on $X$. Let us fix a non-unit $M \in \mathbb{Z}$ and consider the $\operatorname{ring} \Lambda=\mathbb{Z} / M \mathbb{Z}$. We would like to understand when the decomposition holds for $R f_{*} \Lambda_{\tilde{X}}$.

For each stratum $X_{\lambda}$ and point $x \in X_{\lambda}$ we can choose a normal slice $N$ to the stratum $X_{\lambda}$ through $x$. If we set $F:=f^{-1}(x)$ and $\widetilde{N}:=f^{-1}(N)$ we have a commutative diagram of Cartesian squares:


Set $d:=\operatorname{dim} \tilde{N}=\operatorname{dim} N=\operatorname{codim}\left(X_{\lambda} \subset X\right)$. The inclusion $F \hookrightarrow \tilde{N}$ equips the integral homology of $F$ with an intersection form (see [JMW14, § 3.1])

$$
\begin{equation*}
I F_{\lambda}: H_{d-j}(F ; \mathbb{Z}) \times H_{d+j}(F ; \mathbb{Z}) \rightarrow H_{0}\left(\widetilde{N_{\lambda}} ; \mathbb{Z}\right)=\mathbb{Z} \tag{3.1}
\end{equation*}
$$

Remark 3.1. For different points $x, x^{\prime} \in X_{\lambda}$ and normal slices $N, N^{\prime}$ the pairs $f^{-1}(x) \subset f^{-1}(N)$ and $f^{-1}\left(x^{\prime}\right) \subset f^{-1}\left(N^{\prime}\right)$ are diffeomorphic, though not canonically (the isotopy class of diffeomorphism depends on the homotopy type of a path from $x$ to $x^{\prime}$ ).

Let us make the following (restrictive) assumptions, which hold for Schubert varieties and their Bott-Samelson resolutions:
(1) the integral homology $H_{*}(F ; \mathbb{Z})$ of all fibres $F$ of $f$ is free over $\mathbb{Z}$;
(2) each stratum $X_{\lambda}$ is simply connected.

Under these assumptions we have (see [JMW14, § 3]):
Theorem 3.2. The decomposition theorem for $f$ holds with coefficients in $\Lambda$ if and only if all intersection forms (3.1) have the same rank over $\mathbb{Q}$ as they do over $\Lambda$.

The approach of this paper is to calculate these intersection forms in some special cases. In general this is a difficult task. We now consider some situations where the job is easier.

Let $F$ and $\widetilde{N}$ be as above. Suppose first that $\operatorname{dim} F<\frac{1}{2} d$. In this case $H_{d+j}(F)=$ 0 for $j \geq 0$. Hence all intersection forms are zero and the conditions of the theorem are vacuous. If one has the inequality

$$
\operatorname{dim} f^{-1}(x)<\frac{1}{2} \operatorname{codim}\left(X_{\lambda} \subset X\right)
$$

for all strata $X_{\lambda}$ and $x \in X_{\lambda}$ except for those over which $f$ is an isomorphism then $f$ is small. In this case $R f_{*} \Lambda_{X}[\operatorname{dim} X]=\mathbf{I C}(X, \Lambda)$ for any $\Lambda$, which explains why the decomposition theorem is easy in this case.

Remark 3.3. By a theorem of Zelevinsky [Zel83], Schubert varieties in Grassmannians always admit small resolutions. In particular, the stalks and costalks of their integral intersection cohomology complexes are free of $p$-torsion.

Now suppose that $\operatorname{dim} F=\frac{1}{2} d$. In other words $F \subset \tilde{N}$ is half-dimensional and $d$ is the real dimension of $F$. Hence there is only one intersection form which can be non-zero, namely

$$
H_{d}(F ; \mathbb{Z}) \times H_{d}(F ; \mathbb{Z}) \rightarrow \mathbb{Z}
$$

In this case we have a canonical isomorphism

$$
H_{d}(F ; \mathbb{Z})=\bigoplus \mathbb{Z}[Z]
$$

where the direct sum is over the fundamental classes $[Z]$ of the irreducible components $Z \subset F$ of maximal dimension.

If the inequality

$$
\operatorname{dim} f^{-1}(x) \leq \frac{1}{2} \operatorname{codim}\left(X_{\lambda} \subset X\right)
$$

holds for all strata $X_{\lambda}$ and $x \in X_{\lambda}$ then $f$ is called semi-small. In this case there is only one intersection form per stratum. However in this case controlling the intersection forms can be a difficult task.

Example 3.4. A simple and rich source of semi-small maps are provided by the minimal resolutions of Kleinian surface singularities $X$ (i.e. quotients $\mathbb{C}^{2} / \Gamma$ where $\Gamma \subset S L_{2}(\mathbb{C})$ is a finite subgroup). Here $X$ has a unique singular point $0 \in X$ and the exceptional fibre $f^{-1}(0)$ gives a collection of transversely intersecting $\mathbb{P}^{1}$ 's, whose dual graph determines a simply laced Dynkin diagram. The intersection form is given by the negative of the corresponding Cartan matrix. Hence the decomposition theorem is controlled by the determinant of the Cartan matrix. This example has been studied in detail by Juteau [Jut09].
Remark 3.5. In the case of semi-small maps these forms are non-degenerate and even definite (of sign determined by the codimension of the strata). This observation is the starting point for de Cataldo and Migliorini's Hodge theoretic proof of the decomposition theorem [dCM02, dCM05].

Now assume that the equality $\operatorname{dim} F=d$ holds and that $F$ is irreducible. Then $H_{d}(F ; \mathbb{Z})$ is free of rank one (with basis given by the fundamental class $[F]$ ) and the intersection form is a $1 \times 1$-matrix. If $F$ is in addition smooth then we have a diffeomorphism of pairs

$$
(F \subset \tilde{N}) \xrightarrow{\sim}\left(F \subset N_{\tilde{N} / F}\right)
$$

where $N_{\widetilde{N} / F}$ is the normal bundle to $F$ in $\tilde{N}$. It follows from standard algebraic topology that in this case the intersection form is given $p_{!} e\left(N_{\tilde{N} / F}\right)$ where $p_{!}: H^{t o p}(F) \rightarrow \mathbb{Z}$ denotes the trace map on cohomology and $e\left(N_{\widetilde{N} / F}\right)$ denotes the Euler class of the vector bundle $N_{\widetilde{N} / F}$ on $F$.

We refer to this case ( $F$ irreducible and smooth) as the miracle situation because it gives a situation in which the intersection forms are manageable but non-trivial. After all it is not difficult to calculate the determinant of a $1 \times 1$-matrix!

Example 3.6. Suppose that $Y$ is a smooth variety such that $Y \subset T^{*} Y$ may be contracted to a point. (That is, there exists a map $f: T^{*} Y \rightarrow X$ which is an isomorphism on $T^{*} Y \backslash Y$ and maps $Y$ to a point $x_{0} \in X$.) In this case $x_{0}$ is the only singular point in $X$ and $f$ is semi-small. Also, as $f^{-1}\left(x_{0}\right)=Y$ we are in the miracle situation. The intersection form is given by the Euler class of $T^{*} Y$ which is the $-\chi(Y)$, where $\chi(Y)$ denotes the Euler characteristic of $Y$. By the above
discussion, the decomposition theorem holds with coefficients in $\Lambda$ if and only if the image of $\chi(Y)$ in $\Lambda$ is invertible.

An example of this situation is when $Y=\mathbb{P}^{n}$ in which case $X$ may be realized as the space of rank one matrices in $\mathfrak{s l}_{n}(\mathbb{C})$ (a minimal nilpotent orbit), see [JMW12, $\S 3.2$. In this case the intersection form is $\left(-\chi\left(\mathbb{P}^{n}\right)\right)=(-(n+1))$.

## 4. Groups and Schubert varieties

Throughout we work with $G=G L_{N}(\mathbb{C})$, with $T \subset G$ the maximal torus of diagonal matrices. We denote by $W=S_{N}$ the Weyl group of $G$ with simple reflections $S=\left\{s_{i}\right\}_{i=1}^{N-1}$ the simple transpositions. We will often regard $W$ as the subgroup of $G$ of permutation matrices.

Let $\varepsilon_{i}$ denote the character of $T$ given by $\varepsilon_{i}\left(\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{n}\right)\right)=\lambda_{i}$. We let $B$ (resp. $B^{-}$) denote the subgroup of upper (resp. lower) triangular matrices. Let $\Phi=\left\{\varepsilon_{i}-\varepsilon_{j} \mid i \neq j\right\}$ denote the roots, $\Phi^{+}:=\left\{\varepsilon_{i}-\varepsilon_{j} \mid i<j\right\}$ denote the positive roots and $\Phi^{-}:=-\Phi^{+}$the negative roots. For $1 \leq i \leq n$ let $\alpha_{i}:=\varepsilon_{i}-\varepsilon_{i+1}$ denote the simple root.

For any $t=s_{i} \in S$ we denote by $P_{t}$ the minimal standard parabolic subgroup with roots $\Phi^{+} \cup\left\{-\alpha_{i}\right\}$. For any subset $M \subset S$ we consider the corresponding standard parabolic subgroups $W_{M}=\langle t \mid t \in M\rangle \subset W$ and $P_{M}=\left\langle P_{t} \mid t \in M\right\rangle \subset G$. We denote the corresponding subroot system by $\Phi_{M}$ and its positive and negative roots by $\Phi_{M}^{+}$and $\Phi_{M}^{-}$.

For $M \subset S$ we have the partial flag variety $G / P_{M}$. Keeping in mind that we identify $W$ with permutation matrices we have a natural map $W \rightarrow G / B$ whose image is $(G / B)^{T}$, the $T$-fixed points on $G / B$. Simiarly we have a canonical identification

$$
W / W_{M} \xrightarrow{\sim}\left(G / P_{M}\right)^{T} .
$$

We will abuse notation and identify a coset $w W_{M} \in W / W_{M}$ with the corresponding fixed point in $G / P_{M}$. Given any $x W_{M} \in W / W_{M}$ we have a Schubert cell

$$
X_{x}:=B \cdot x P_{M} / P_{M} \subset G / P_{M}
$$

(an affine space) and its closure

$$
\bar{X}_{x} \subset G / P_{M}
$$

a Schubert variety.
Let $\zeta^{\vee}: \mathbb{C}^{*} \rightarrow T$ denote a dominant regular cocharacter (i.e. such that the induced action of $\mathbb{C}^{*}$ on the Lie algebra of the unipotent radical of $B$ has strictly positive weights). Then the Białynicki-Birula cells on $G / P_{M}$ coincide with the Bruhat cells. In other words, for any for any $T$-fixed point $x W_{M}$ in $G / P_{M}$ we have:

$$
X_{x}=\left\{q \in G / P_{M} \mid \lim _{z \rightarrow 0} \zeta^{\vee}(z) \cdot q=x\right\}
$$

If instead we consider the dual Białynicki-Birula cells we get a stratification dual to the Bruhat stratification. We have

$$
S_{x}:=B^{-} \cdot x P_{M} / P_{M}=\left\{q \in G / P_{M} \mid \lim _{z \rightarrow \infty} \zeta^{\vee}(z) \cdot q=x\right\}
$$

## 5. Bott-Samelson varieties

Given a sequence $\underline{w}:=t_{1} t_{2} \ldots t_{m}$ with $t_{i} \in S$ consider the Bott-Samelson variety

$$
B S(\underline{w}):=P_{t_{1}} \times{ }_{B} P_{t_{2}} \times{ }_{B} \cdots \times_{B} P_{t_{m}} / B
$$

defined as the quotient of $P_{t_{1}} \times P_{t_{2}} \times \cdots \times P_{t_{m}}$ by $B^{m}$ acting on the right by

$$
\left(p_{1}, p_{2} \ldots, p_{m}\right) \cdot\left(b_{1}, b_{2}, \ldots, b_{m}\right)=\left(p_{1} b_{1}, b_{1}^{-1} p_{2} b_{2}, \ldots, b_{m-1}^{-1} p_{m} b_{m}\right)
$$

We denote the image of $\left(p_{1}, \ldots, p_{m}\right)$ in $B S(\underline{w})$ by $\left[p_{1}, \ldots, p_{m}\right]$. The Bott-Samelson variety $B S(\underline{w})$ is a (left) $B$-variety via $b \cdot\left[p_{1}, p_{2}, \ldots, p_{m}\right]:=\left[b p_{1}, p_{2}, \ldots, p_{m}\right]$.

Given any subexpression $\underline{e}$ of $\underline{w}$ we have a point $[\underline{e}]:=\left[s_{1}^{e_{1}}, \ldots, s_{m}^{e_{m}}\right] \in B S(\underline{w})$ which is fixed by $T$. For any $t \in S$ we let $u_{t}(\lambda)$ denote the root subgroup corresponding to $-\alpha_{t}$. Then for any subexpression $\underline{e}$ we have a chart around [ $\left.\underline{e}\right]$ given by

$$
\begin{equation*}
\mathbb{C}^{m} \ni\left(\lambda_{1}, \ldots, \lambda_{m}\right) \mapsto\left[t_{1}^{e_{1}} u_{t_{1}}\left(\lambda_{1}\right), t_{2}^{e_{2}} u_{t_{2}}\left(\lambda_{2}\right), \ldots, t_{m}^{e_{m}} u_{t_{m}}\left(\lambda_{m}\right)\right] \in B S(\underline{w}) . \tag{5.1}
\end{equation*}
$$

We denote this chart by $C_{\underline{e}} \subset B S(\underline{w})$. The charts $C_{\underline{e}}$ cover $B S(\underline{w})$ as we run over all subexpressions $\underline{e}$. Moreover, using the relation $\gamma u_{t}(\lambda) \gamma^{-1}=u_{t}\left(\alpha_{t}(\gamma)^{-1} \lambda\right)$ one checks easily that the $T$-action on $C_{\underline{e}}$ is linear, with weights

$$
\begin{equation*}
\left(t_{1}^{e_{1}}\left(-\alpha_{t_{1}}\right),\left(t_{1}^{e_{1}} t_{2}^{e_{2}}\right)\left(-\alpha_{t_{2}}\right), \ldots,\left(t_{1}^{e_{1}} \ldots t_{m}^{e_{m}}\right)\left(-\alpha_{t_{m}}\right)\right) \tag{5.2}
\end{equation*}
$$

In particular the set $\{[\underline{e}] \mid \underline{e}$ a subexpression of $\underline{w}\}$ coincides with the set of $T$-fixed points on $B S(\underline{w})$.

For any subsequence $\underline{e}$ of $\underline{w}$ we have a closed subvariety

$$
B S(\underline{e}):=\left\{\left[p_{1}, \ldots, p_{m}\right] \mid p_{i}=1 \text { if } e_{i}=0\right\} \subset B S(\underline{w}) .
$$

For example $B S(11 \ldots 1)=B S(\underline{w})$ and $B S(00 \ldots 0)=\mathrm{pt}$. It is easy to see that $B S(\underline{e})$ is isomorphic to the Bott-Samelson variety $B S(\underline{z})$ where $\underline{z}$ is the expression obtained from $s_{1}^{e_{1}} \ldots s_{m}^{e_{m}}$ by deleting all occurrences of the identity.

We denote by $C_{\underline{e}}^{+} \subset B S(\underline{w})$ the Białynicki-Birula cell corresponding to the $T$ fixed poing $[\underline{e}]$ and cocharacter $\zeta^{\vee}$. That is

$$
C_{\underline{e}}^{+}:=\left\{x \in B S(\underline{w}) \mid \lim _{z \rightarrow 0} \zeta^{\vee}(z) \cdot x=[\underline{e}]\right\}
$$

Because $C_{\underline{e}} \subset B S(\underline{w})$ is open and $T$-stable we have

$$
C_{\underline{e}}^{+} \subset C_{\underline{e}}
$$

Moreover, from the above calculation of $T$-weights it follows that

$$
\begin{equation*}
C_{\underline{e}}^{+}=\left\{\left(\lambda_{i}\right) \in C_{\underline{e}} \mid \lambda_{i}=0 \text { if }\left(t_{1}^{e_{1}} \ldots t_{i}^{e_{i}}\right)\left(-\alpha_{t_{i}}\right) \in \Phi^{-}\right\} \tag{5.3}
\end{equation*}
$$

(We use the above identification of $C_{\underline{e}}$ with $\mathbb{C}^{m}$.)
For any $M \subset S$, the multiplication map induces a proper morphism of varieties

$$
f: B S(\underline{w}) \rightarrow G / P_{M}
$$

If $w$ is minimal in its coset $w W_{M}$ and if $\underline{w}$ is a reduced expression for $w$ then $f$ is an isomorphism over the Schubert cell $X_{w} \subset G / P_{M}$. In particular, in this case $f$ gives a resolution of singularities of the Schubert variety $\bar{X}_{w}$. The following easy lemma will be useful later:

Lemma 5.1. The $T$-fixed points in the fibre $f^{-1}\left(x W_{M}\right)$ are given by

$$
\left\{[\underline{e}] \mid \text { subexpressions } \underline{e} \text { of } \underline{w} \text { with }\left(\underline{w}^{\underline{e}}\right) W_{M}=x W_{M}\right\}
$$

## 6. Combinatorics of reduced expressions

In this section we define the reduced expression which determines both the Schubert variety and the Bott-Samelson resolution occurring in Theorem 1.3. We also establish two combinatorial lemmas involving this subexpression. Their statements are essentially copied from [Wil13].

Remark 6.1. The reduced expression combinatorics involved in [Wil13] is slightly more general than that considered here. We hope that this makes the treatment below easier to follow.

Recall the ring $R=\mathbb{Z}\left[\varepsilon_{1}, \ldots, \varepsilon_{n}\right]$, the divided difference operators $\partial_{i}: R \rightarrow R$ and their composites $\partial_{w}: R \rightarrow R$ for $w \in S_{n}$ from the introduction. Fix an expression of the form:

$$
C=\partial_{w_{m}}\left(\varepsilon_{n}^{a_{m}} \ldots \partial_{w_{2}}\left(\varepsilon_{n}^{a_{2}} \partial_{w_{1}}\left(\varepsilon_{n}^{a_{1}}\right)\right) \ldots\right)
$$

The following assumption will be in place for the rest of this paper:

$$
\begin{equation*}
0 \neq C \in \mathbb{Z} \tag{6.1}
\end{equation*}
$$

We set $a:=\sum_{i=1}^{m} a_{i}$. Because $\varepsilon_{i}$ has degree 2 and $\partial_{w}$ has degree $-2 \ell(w),(6.1)$ is equivalent to the assumptions:

$$
\begin{gather*}
a=\sum_{i=1}^{m} \ell\left(w_{i}\right),  \tag{6.2}\\
C \neq 0 \tag{6.3}
\end{gather*}
$$

Because $\left[\partial_{j}, \varepsilon_{n}\right]=0$ for $j \neq n-1$ we may and do assume that $w_{i}$ is minimal in $S_{n} /\left\langle s_{1}, \ldots, s_{n-2}\right\rangle$ for all $i$. It follows that each $w_{i}$ has a unique reduced expression. It has the form

$$
\underline{w}_{i}=s_{k_{i}} s_{k_{i}+1} \ldots s_{n-1}
$$

for some $1 \leq k_{i} \leq n-1$.
We work in $S_{N}$ where $N=n+a$. Consider the subsets $M=\left\{s_{1}, \ldots, s_{n-1}\right\}$ and $A=\left\{s_{n+1}, \ldots, s_{n+a-1}\right\}$. That is, we divide the nodes of our Coxeter diagram as follows:


Let $W_{M}$ and $W_{A}$ denote the corresponding parabolic subgroups. The simple reflection $s_{n}$ plays a special role, as will become clear shortly.

Consider

$$
\underline{x}^{=} \underline{w}_{m} \underline{z}_{m} \cdots \underline{w}_{2} \underline{z}_{2} \underline{w}_{1} \underline{z}_{1}
$$

where

$$
\begin{aligned}
\underline{z}_{1}= & \left(s_{n} s_{n+1} \ldots s_{n+a_{1}-1}\right) \ldots\left(s_{n} s_{n+1}\right)\left(s_{n}\right) \\
\underline{z}_{2}= & \left(s_{n} s_{n+1} \ldots s_{n+a_{1}+a_{2}-1}\right) \ldots\left(s_{n} s_{n+1} \ldots s_{n+a_{1}+1}\right)\left(s_{n} s_{n+1} \ldots s_{n+a_{1}}\right) \\
& \vdots \\
\underline{z}_{m}= & \left(s_{n} s_{n+1} \ldots s_{b-1}\right) \ldots\left(s_{n} s_{n+1} \ldots s_{n+a-a_{m}+1}\right)\left(s_{n} s_{n+1} \ldots s_{n+a-a_{m}}\right) .
\end{aligned}
$$

We denote by $\underline{z}_{1}^{\prime}, \underline{z}_{2}^{\prime}, \ldots, \underline{z}_{m}^{\prime}$ the similar reduced expressions with all occurrences of $s_{n}$ deleted:

$$
\begin{aligned}
\underline{z}_{1}^{\prime}= & \left(s_{n+1} \ldots s_{n+a_{1}-1}\right) \ldots\left(s_{n+1}\right) \\
& \vdots \\
\underline{z}_{m}^{\prime}= & \left(s_{n+1} \ldots s_{a-1}\right) \ldots\left(s_{n+1} \ldots s_{n+a-a_{m}+1}\right)\left(s_{n+1} \ldots s_{n+a-a_{m}}\right)
\end{aligned}
$$

Remark 6.2. The expression $\underline{z}_{m} \cdots \underline{z}_{2} \underline{z}_{1}$ is a reduced expression for $w_{\left\{s_{n}\right\} \cup A}$. Similarly, $\underline{z}_{m}^{\prime} \cdots \underline{z}_{2}^{\prime} \underline{z}_{1}^{\prime}$ is a reduced expression for $w_{A}$.

Example 6.3. We give an example of the expression $\underline{x}$. Let $n=4$ and consider $\underline{w}_{i}$ defined as follows:

$$
\underline{w}_{1}=s_{2} s_{3}, \quad \underline{w}_{2}=s_{3}, \quad \underline{w}_{3}=s_{2} s_{3}, \quad \underline{w}_{4}=s_{1} s_{2} s_{3}=\underline{w}_{5} .
$$

Take $a_{1}=3, a_{2}=2, a_{3}=2, a_{4}=2, a_{5}=2$, so that $a=3+2+2+2+2=11$ and $N=4+11=15$. Then we may depict $\underline{x}$ via the following string diagram:


Let $x$ denote the element of $W$ expressed by $\underline{x}$. The following lemma follows by careful consideration of a string diagram depicting $\underline{x}$ (see Example 6.3 above):

Lemma 6.4. (1) $\underline{x}$ is a reduced expression for $x$;
(2) $x$ is minimal in its coset $x W_{M}$.

Write $\underline{x}=t_{1} \ldots t_{\ell}$.
Lemma 6.5. Any subsequence $\underline{e}$ of $\underline{x}$ with $\underline{x} \underline{e} \in w_{A} W_{M}$ has $\varepsilon_{i}=1$ if $t_{i} \in A$ and $\varepsilon_{i}=0$ if $t_{i}=s_{n}$.

Proof. This is (a special case of) Lemma 5.6 in [Wil13]. We give an idea of the proof: We have already remarked that $\underline{z}_{m}^{\prime} \cdots \underline{z}_{2}^{\prime} \underline{z}_{1}^{\prime}$ is a reduced expression for $\underline{w}_{A}$. In particular, to achieve $\underline{x}^{\underline{e}} \in w_{A} W_{M}$ we must have $\varepsilon_{i}=1$ for every $i$ with $t_{i} \in A$. Now, if $\varepsilon_{i}=1$ for some $i$ with $t_{i}=s_{n}$ then it is impossible for $\underline{x} \underline{e}$ to belong to $W_{M \cup A}$, as is seen by considering the string diagram of $\underline{x}$. The result follows.

## 7. Geometry

We keep the notation from the previous section. Let $G=G L_{N}(\mathbb{C})$ and $P_{M}$ denote the standard parabolic subgroup corresponding to $M=\left\{s_{1} \ldots, s_{n-1}\right\}$.

Consider the Bott-Samelson variety associated to the expression $\underline{x}=t_{1} \ldots t_{\ell}$ defined in the previous section:

$$
\operatorname{BigBS}:=B S(\underline{x})=P_{t_{1}} \times_{B} P_{t_{2}} \times_{B} \cdots \times_{B} P_{t_{\ell}} \times_{B} P_{M} / P_{M}
$$

It follows from Lemma 6.4 that multiplication induces a resolution of singularities

$$
f: \operatorname{BigBS} \rightarrow X_{x}
$$

where $X_{x}$ denotes the Schubert variety $X_{x}:=\overline{B x P_{M} / P_{M}} \subset G / P_{M}$.
Consider the closed subvariety

$$
F:=\left\{\left[g_{1}, \ldots, g_{\ell}\right] \in \operatorname{BigBS} \mid g_{i}=1 \text { if } t_{i}=s_{n}, g_{i}=t_{i} \text { if } t_{i} \in A\right\} \subset \operatorname{BigBS}
$$

(More precisely, we consider the image of the corresponding subset in $P_{t_{1}} \times P_{t_{2}} \times$ $\cdots \times P_{t_{\ell}}$ in BigBS.)

Lemma 7.1. $F=f^{-1}\left(w_{A} P_{M} / P_{M}\right)$.
Proof. It is clear that $F \subset f^{-1}\left(w_{A} P_{M} / P_{M}\right)$. It remains to show the reverse inclusion. Consider the subexpression $\underline{g}:=g_{1} \ldots g_{\ell}$ where $g_{i}=1$ if $t_{i} \in M \cup A$ and $g_{i}=0$ if $t_{i}=s_{n}$. We first claim that

$$
\begin{equation*}
f^{-1}\left(w_{A} P_{M} / P_{M}\right) \subset B S(\underline{g}) \tag{7.1}
\end{equation*}
$$

(The subvariety $B S(\underline{g})$ was defined in $\S 5$.) Suppose that $q \in f^{-1}\left(w_{A} P_{M} / P_{M}\right)$. Recall our dominant regular cocharacter $\zeta^{\vee}$ from earlier. Then $\lim _{z \rightarrow 0} \zeta^{\vee}(z) \cdot q \in$ $f^{-1}\left(w_{A} P_{M} / P_{M}\right)$ and is a $T$-fixed point. Hence $\lim _{z \rightarrow 0} \zeta^{\vee}(z) \cdot q=[\underline{e}]$ for some subexpression $\underline{e}$ of $\underline{x}$, and $q \in C_{\underline{e}}^{+}$. Now combining Lemma 5.1 and Lemma 6.5, we see that $\underline{e}$ satisfies $e_{i}=0$ if $^{-} t_{i}=s_{n}$ and $e_{i}=1$ if $t_{i} \in A$. It follows that $C_{\underline{e}}^{+} \subset B S(\underline{g})$ by (5.3). (We use that if $\alpha \in \Phi^{-}$is not in $\Phi_{M \cup A}$, then $w(\alpha) \in \Phi^{-}$ for all $\left.w \in \bar{W}_{M \cup A}.\right)$ Now (7.1) follows.

Because the sets $M$ and $A$ are disconnected in the Dynkin diagram we have an isomorphism

$$
\begin{equation*}
B S(\underline{g}) \xrightarrow{\sim} B S\left(\underline{w}_{m} \underline{w}_{m-1} \cdots \underline{w}_{1}\right) \times B S\left(\underline{z}_{m}^{\prime} \underline{z}_{m-1}^{\prime} \cdots \underline{z}_{1}^{\prime}\right) \tag{7.2}
\end{equation*}
$$

which commutes with the multiplication map $f$. (The expressions $\underline{z}_{i}^{\prime}$ were defined in the previous section.) On the first factor of the right hand side of (7.2) the multiplication map is the projection to the base point $P_{M} / P_{M} \in G / P_{M}$. Now $\underline{z}_{m}^{\prime} \underline{z}_{m-1}^{\prime} \cdots \underline{z}_{1}^{\prime}$ is a reduced expression for $w_{A}$, and hence the fibre of

$$
B S\left(\underline{z}_{m}^{\prime} \underline{z}_{m-1}^{\prime} \cdots \underline{z}_{1}^{\prime}\right) \rightarrow G / P_{M}
$$

over $w_{A}$ consists only of one point. The result follows.
The following is easily deduced from (7.2).

Corollary 7.2. The fibre $F$ is smooth and we have a $T$-equivariant isomorphism

$$
\phi: B S\left(\underline{w}_{m} \cdots \underline{w}_{1}\right) \xrightarrow{\sim} F .
$$

In particular, by (6.2):

$$
\begin{equation*}
\operatorname{dim} F=a \tag{7.3}
\end{equation*}
$$

Recall the dual cells $X_{w_{A}} \subset G / P_{M}$ and $S_{w_{A}} \subset G / P_{M}$ which are the attracting and repelling sets for the fixed point $w_{A} \in G / P_{M}$ and cocharacter $\zeta^{\vee}: \mathbb{C}^{*} \rightarrow T$. Because $S_{w_{A}}$ is a normal slice to the stratum $X_{w_{A}} \subset \bar{X}_{x} \subset G / P_{M}$ we conclude:

$$
\begin{gather*}
f^{-1}\left(S_{w_{A}}\right) \subset \operatorname{BigBS} \text { is smooth }  \tag{7.4}\\
\operatorname{dim} f^{-1}\left(S_{w_{A}}\right)=\operatorname{dim} S_{w_{A}}=\ell(x)-\ell\left(w_{A}\right)=2 a \tag{7.5}
\end{gather*}
$$

In particular we are in the "miracle situation": by (7.3) and Lemma 7.1 the fibre of $f$ over $w_{A} P_{M} / P_{M}$ is irreducible, smooth, and half-dimensional inside $f^{-1}\left(S_{w_{A}}\right)$.

By the discussion in $\S 3$ it follows that in order to decide when the decomposition theorem holds at $w_{A}$ we need to calculated the self-intersection of $F$ inside $f^{-1}\left(S_{w_{A}}\right)$. This will be done in the next section, and will rely on the following lemma.

Lemma 7.3. Let [e] be a T-fixed point belonging to $F$. Then the $T$-weights on the normal bundle to $F \subset f^{-1}\left(S_{w_{A}}\right)$ at [e] are

$$
\left\{t_{1}^{e_{1}} t_{2}^{e_{2}} \ldots t_{j}^{e_{j}}\left(-\alpha_{n}\right) \mid 1 \leq j \leq \ell ; t_{j}=s_{n}\right\}
$$

Proof. Consider the chain of inclusions (where $T_{[\underline{e}]}$ denotes the tangent space)

$$
T_{[e]} F \subset T_{[\underline{ }]}\left(f^{-1}\left(S_{w_{A}}\right)\right) \subset T_{[e]} \mathrm{BigBS}
$$

Our goal is to calculate the $T$-weights on the normal bundle to $F \subset f^{-1}\left(S_{w_{A}}\right)$ at the $T$-fixed point $[\underline{e}]$ :

$$
\left(N_{F}\left(f^{-1} S_{w_{A}}\right)\right)_{[e]}=T_{[e]}\left(f^{-1} S_{w_{A}}\right) / T_{[e]} F .
$$

We work in the chart $C_{\underline{e}}$ around $[\underline{e}]$. As $C_{\underline{e}}$ is an affine space with linear $T$-action we have a $T$-equivariant identification $C_{\underline{e}}=T_{[\underline{e}]} B S(\underline{x})$. Under this identification we claim:

$$
\begin{gather*}
T_{[\ell]} F=\left\{\left(\lambda_{i}\right)_{i=1}^{\ell} \mid \lambda_{i}=0 \text { unless } t_{i} \in W_{M}\right\},  \tag{7.6}\\
T_{[e]} S_{w_{A}}=\left\{\left(\lambda_{i}\right)_{i=1}^{\ell} \mid \lambda_{i}=0 \text { if } t_{i} \in W_{A}\right\} \tag{7.7}
\end{gather*}
$$

(We identify $C_{\underline{e}}=\mathbb{C}^{\ell}$ as in (5.1).) The first equality follows from the proof of the previous lemma. For the second equality notice that if $j$ is such that $t_{j}=s_{n}$ (and hence $e_{j}=0$ ) then the curve

$$
c: \gamma \mapsto\left[s_{1}^{e_{1}}, \ldots, s_{j-1}^{e_{j-1}}, u_{\alpha_{n}}(\gamma), s_{j+1}^{e_{j+1}}, \ldots, s_{m}^{e_{m}}\right] \in \operatorname{BigBS}
$$

has $T$-weight in $\left.\Phi^{-} \backslash \Phi_{M \cup A}^{-}\right)$. (Recall that $e_{j}=0$ if $t_{j}=s_{n}$ and that $w\left(-\alpha_{n}\right) \in \Phi^{-} \backslash$ $\Phi_{M \cup A}^{-}$for all $w \in W_{M \cup A}$.) In particular for any $\gamma \in \mathbb{C}, \lim _{z \rightarrow \infty} \zeta^{\vee}(z) \cdot c(\gamma)=[\underline{e}]$. From the definition of $S_{w_{A}}$ as a repelling set we deduce that $f \circ c$ is contained in $S_{w_{A}}$ and hence the image of $c$ is contained in $f^{-1}\left(S_{w_{A}}\right)$. Taking derivatives of all such curves we deduce an inclusion

$$
\begin{equation*}
\left\{\left(\lambda_{i}\right)_{i=1}^{\ell} \mid \lambda_{i}=0 \text { if } t_{i} \in W_{A}\right\} \subset T_{[e]} S_{w_{A}} . \tag{7.8}
\end{equation*}
$$

However both sides have dimension $2 a$ : the left hand side by inspection, and the right hand side by (7.5). We deduce that (7.8) is an equality, which is (7.7).

The lemma now follows easily from (7.6), (7.7) and (5.2).

## 8. Euler class lemma

The goal of this section is to prove a lemma which computes the proper direct image of certain "combinatorial" cohomology classes in the equivariant cohomology of Bott-Samelson resolutions.

We keep the notation of the previous sections. Recall that $T \subset G=G L_{n}(\mathbb{C})$ denotes the maximal torus of diagonal matrices and $\varepsilon_{i}$ for $1 \leq i \leq n$ denote the coordinate characters. Given a $T$-space $X$ we denote by $H_{T}^{*}(X)$ its equivariant cohomology. (In this section we always take cohomology with coefficients in $\mathbb{Z}$.) The Borel isomorphism gives a canonical isomorphism

$$
H_{T}^{*}(p t)=\mathbb{Z}\left[\varepsilon_{1}, \ldots, \varepsilon_{n}\right]=R
$$

with $\operatorname{deg} \varepsilon_{i}=2$ for $1 \leq i \leq m$.
Let us fix an expression $\underline{w}=t_{1} t_{2} \ldots t_{m}$ and let $B S(\underline{w})$ denote the corresponding Bott-Samelson variety. By the localization theorem the restriction map

$$
H_{T}^{*}(B S(\underline{w})) \rightarrow H_{T}^{*}\left(B S(\underline{w})^{T}\right)=\bigoplus_{\underline{e} \subset \underline{w}} H_{T}^{*}([\underline{e}])
$$

is injective. In particular, any cohomology class $c \in H_{T}^{*}(B S(\underline{w})$ is determined by a tuple ( $c_{\underline{e}}$ ) of elements of $R$ indexed by all subexpressions $\underline{e}$ of $\underline{w}$.

We say that a class $c \in H_{T}^{*}(B S(\underline{w})$ is combinatorial if there exists polynomials $f_{1}, f_{2}, \ldots, f_{m}$ such that, for any subexpression $\underline{e}=e_{1} \ldots e_{m}$ of $\underline{w}$, we have

$$
c_{\underline{e}}=s_{1}^{e_{1}}\left(f_{1} s_{2}^{e_{2}}\left(f_{2} \ldots s_{m}^{e_{m}}\left(f_{m}\right) \ldots\right)\right)
$$

Given a combinatorial $c$ we say that it is described by the polynomials $f_{1}, f_{2}, \ldots, f_{m}$.
Example 8.1. We give an example of a naturally occurring combinatorial cohomology class. Fix a representation $V$ of $B^{m}$ and consider the induced bundle

$$
L_{V}:=\left(P_{t_{1}} \times \cdots \times P_{t_{m}}\right) \times_{B^{m}} V
$$

which is naturally a vector bundle on $B S(\underline{w})$ with fibre $V$. For any $1 \leq i \leq m$ let $V_{i}$ denote the restriction of $V$ to the $i^{t h}$ copy $B \subset B^{m}$ and let $f_{i}:=\operatorname{det} V_{i} \in$ $\mathbb{Z}\left[\varepsilon_{1}, \ldots, \varepsilon_{m}\right]$ denote the product of the characters of $T \subset B$ occurring in $V_{i}$. Then the equivariant Euler class of $L_{V}$ is combinatorial, being described by the polynomials $f_{1}, f_{2}, \ldots, f_{m}$.

Let $\underline{v}:=t_{1} t_{2} \ldots t_{m-1}$ be the expression obtained by ignoring the last term of $\underline{w}$. The projection map $P_{t_{1}} \times \cdots \times P_{t_{m-1}} \times P_{t_{m}} \rightarrow P_{t_{1}} \times \cdots \times P_{t_{m-1}}$ induces a morphism $r: B S(\underline{w}) \rightarrow B S(\underline{v})$
which is easily seen to be a $\mathbb{P}^{1}$-fibration. Given a subexpression $\underline{e}$ of $\underline{v}$ we obtain two subexpressions of $\underline{w}$ by appending either a 0 or a 1 to $\underline{e}$. We denote these subexpressions simply by $\underline{e} 0$ and $\underline{e} 1$. The $T$-fixed points in the fibre $r^{-1}([\underline{v}])$ are precisely the points $\underline{e} 0$ and $\underline{e} 1$.
Proposition 8.2. Suppose that $c \in H_{T}^{*}(B S(\underline{w})$ is a combinatorial class described by $f_{1}, f_{2}, \ldots, f_{m}$. Then $r_{!}(c)$ is also combinatorial and is described by $g_{1}, \ldots, g_{m-1}$ where

$$
g_{i}:= \begin{cases}f_{i} & \text { if } i<m-1 \\ f_{m-1} \partial_{t_{m}}\left(f_{m}\right) & \text { if } i=m-1\end{cases}
$$

The following well-known lemma provides the key calculation:
Lemma 8.3. Suppose that $X=\mathbb{P}^{1}$ with non-trivial linear $T$-action and weights at 0 and $\infty$ given by $-\gamma$ and $\gamma$ respectively. For any class $g=\left(g_{0}, g_{\infty}\right) \in H_{T}^{*}\left(\mathbb{P}^{1}\right)$ we have

$$
p_{!}(g)=\frac{g_{0}-g_{\infty}}{\gamma}
$$

where $p: X \rightarrow \mathrm{pt}$ is the projection.
Proof. By the localization theorem $H_{T}^{*}\left(\mathbb{P}^{1}\right)$ identifies with pairs $\left(g_{0}, g_{\infty}\right) \in R \oplus R$ such that $g_{0}-g_{\infty} \in(\gamma)$. It is easy to see that it is free over $R$ with generators in degree 0,2 . Hence $p_{!}$is determined by what what it does to the $R$-basis $(1,1)$ and $(0, \gamma)$. However it must annihilate $(1,1)$ for degree reasons and must send $(-\gamma, \gamma)$ to 2 (the Euler characteristic of $\mathbb{P}^{1}$ ). Hence $p$ ! must be given by the above formula.

Proof of Proposition 8.2. We claim that the localization of the push-forward of $c$ at the point $[\underline{e}]$ (for $\underline{e}$ a subexpression of $\underline{v}$ ) is given by

$$
\begin{equation*}
\left(r_{!} c\right)_{\underline{e}}=\frac{c_{\underline{e} 0}-c_{\underline{e} 1}}{t_{1}^{e_{1}} \ldots t_{m-1}^{e_{m-1}}\left(\alpha_{t_{m}}\right)} \tag{8.1}
\end{equation*}
$$

As remarked above, $f^{-1}([\underline{v}])$ is isomorphic to $\mathbb{P}^{1}$. Moreover, by (5.2) the $T$-weights at the $T$-fixed points $\underline{e} 0$ and $\underline{e} 1$ are $t_{1}^{e_{1}} \ldots t_{m-1}^{e_{m-1}}\left(-\alpha_{t_{m}}\right)$ and $t_{1} \ldots t_{m-1}^{e_{m-1}}\left(\alpha_{t_{m}}\right)$ respectively. The equality in (8.1) now follows by Lemma 8.3 and proper base change.

By our assumption that $c$ is combinatorial and described by $f_{1}, \ldots, f_{m}$ we can rewrite the right hand side of (8.1) as

$$
t_{1}^{e_{1}}\left(f_{1} \ldots t_{m-1}^{e_{m-1}}\left(f_{m-1}\left(\frac{f_{m}-t_{m} f_{m}}{\alpha_{t_{m}}}\right)\right) \ldots\right)=t_{1}^{e_{1}}\left(f_{1} \ldots t_{m-1}^{e_{m-1}}\left(f_{m-1} \partial_{t_{m}}\left(f_{m}\right)\right) \ldots\right)
$$

which is what we wanted to show.
By iterating the above proposition to the maps $B S(\underline{w}) \rightarrow B S(\underline{v}) \rightarrow \ldots \rightarrow \mathrm{pt}$ we deduce:

Corollary 8.4. Let $c \in H_{T}^{*}(B S(\underline{w})$ be a combinatorial class described by

$$
f_{1}, f_{2}, \ldots, f_{m}
$$

Then

$$
p_{!}(c)=\partial_{t_{1}}\left(f_{1} \partial_{t_{2}}\left(f_{2} \ldots \partial_{t_{m}}\left(f_{m}\right) \ldots\right)\right)
$$

where $p: B S(\underline{w}) \rightarrow$ pt denotes the projection.
Remark 8.5. Corollary 8.4 seems to be very useful for calculating the proper pushforward of Euler classes of vector bundles on Bott-Samelson varieties (see Example 8.1). This explains the title of this section. In the next section we will see another example of its utility.

## 9. Proof of the main theorem

Finally, we return to the setting of $\S 6$ and $\S 7$. Recall our reduced expression $\underline{x}$ for $s \in S_{N}$, our Schubert variety $\bar{X}_{x} \subset G / P_{M}$, our resolution of singularities

$$
f: \operatorname{BigBS} \rightarrow \bar{X}_{x}
$$

and the normal slice $S_{w_{A}} \subset G / P_{M}$ to the Schubert cell $X_{w_{A}} \subset G / P_{M}$. We saw in $\S 7$ that we are in the miracle situation. Namely, that the fibre $F:=f^{-1}\left(w_{A}\right)$ is smooth and irreducible, and is half-dimensional inside $f^{-1}\left(S_{w_{A}}\right)$.

Moreover we saw in Corollary 7.2 that we have a $T$-equivariant isomorphism

$$
\phi: B S\left(\underline{w}_{m} \cdots \underline{w}_{1}\right) \xrightarrow{\sim} F .
$$

Let us define polynomials $f_{i} \in R$ for $i=1, \ldots, \sum \ell\left(w_{i}\right)=a$ by

$$
\begin{gathered}
f_{1}=f_{2}=\cdots=f_{\ell\left(w_{m}\right)-1}=1, \quad f_{\ell\left(w_{m}\right)}=\left(\varepsilon_{n}-\varepsilon_{n+1}\right) \ldots\left(\varepsilon_{n}-\varepsilon_{n+a_{m}}\right), \\
f_{\ell\left(w_{m}\right)+1}=\cdots=f_{\ell\left(w_{m}\right)+\ell\left(w_{m-1}\right)-1}=1, \quad f_{\ell\left(w_{m}\right)+\ell\left(w_{m-1}\right)}=\left(\varepsilon_{n}-\varepsilon_{n+1}\right) \ldots\left(\varepsilon_{n}-\varepsilon_{n+a_{m-1}}\right), \\
\vdots \\
\vdots \\
f_{\ell\left(w_{m}\right)+\ldots \ell\left(w_{2}\right)+1}=\cdots=f_{a-1}=1, \quad f_{a}=\left(\varepsilon_{n}-\varepsilon_{n+1}\right) \ldots\left(\varepsilon_{n}-\varepsilon_{n+a_{1}}\right) .
\end{gathered}
$$

Lemma 9.1. Under the isomorphism $\phi$ above the equivariant Euler class of the normal bundle of $F \subset f^{-1}\left(S_{w_{A}}\right)$ is combinatorial, and is described by the polynomials $f_{1}, \ldots, f_{a}$.
Proof. Recall that the localization in $T$-equivariant cohomology of an Euler class of an equivariant vector bundle is given by the product of the $T$-weights at each fixed point. Hence, by Lemma 7.3 the localization of the Euler class of the normal bundle at a $T$-fixed point $\underline{e}$ in $F$ is given by the product

$$
\begin{equation*}
E_{\underline{e}}:=\prod_{\substack{1 \leq j \leq \ell \\ t_{j}=s_{n}}} t_{1}^{e_{1}} \cdots t_{j}^{e_{j}}\left(-\alpha_{n}\right) \tag{9.1}
\end{equation*}
$$

To complete the proof, we will argue that we can rewrite the above expression to yield a combinatorial class described by the above polynomials.

Recall that $\underline{x}$ has the form

$$
\underline{x}^{=} \underline{w}_{m} \underline{z}_{m} \cdots \underline{w}_{2} \underline{z}_{2} \underline{w}_{1} \underline{z}_{1}
$$

where each $\underline{w}_{i}$ is an expression in $M$, and each $\underline{z}_{i}$ an expression in $A \cup\left\{s_{n}\right\}$. Let us write

$$
\underline{e}=\underline{e}_{m}^{\prime} \underline{f}_{m}^{\prime} \cdots \underline{e}_{2}^{\prime} \underline{f}_{2}^{\prime} \underline{e}_{1}^{\prime} \underline{f}_{1}^{\prime},
$$

where each $\underline{e}_{i}^{\prime}$ (resp. $\underline{f}_{i}^{\prime}$ ) is the corresponding subexpression of $\underline{w}_{i}$ (resp. $\underline{z}_{i}$ ). By Lemma 6.5 the fixed points $[\underline{e}]$ in $\operatorname{BigBS}$ correspond to those subexpressions $\underline{e}$ of $\underline{x}$ with $e_{i}=0$ if $t_{i}=s_{n}$ and $e_{1}=1$ if $t_{i} \in A$. An alternative way of saying this is that we have no choice for the subexpressions $\underline{f}_{i}^{\prime}$ : if we bracket $\underline{z}_{i}$ as

$$
\left(s_{n} s_{n+1} \ldots s_{n+a_{1}+\cdots+a_{i}-1}\right) \ldots\left(s_{n} s_{n+1} \ldots s_{n+a_{1}+\cdots+a_{i-1}+1}\right)\left(s_{n} s_{n+1} \ldots s_{n+a_{1}+\cdots+a_{i-1}}\right)
$$

then $\underline{f}_{i}^{\prime}$ has the form

$$
(01 \ldots 1) \ldots(01 \ldots 1)(01 \ldots 1)
$$

Hence we can rewrite $E_{\underline{e}}$ as

$$
\underline{w}_{m}^{\underline{e}_{m}^{\prime}}\left(n_{m}\right) \cdot \underline{w}_{m}^{e_{m}^{\prime}} \underline{w}_{m-1}^{\underline{e}_{m-1}^{\prime}}\left(n_{m-1}\right) \cdot \quad \ldots \quad \cdot \underline{w}_{m}^{e_{m}^{\prime}} \underline{w}_{m-1}^{\underline{e}_{m-1}^{\prime}} \cdots \underline{w}_{1}^{e_{1}^{\prime}}\left(n_{1}\right)
$$

where

$$
n_{i}=\left(\varepsilon_{n}-\varepsilon_{n+1}\right)\left(\varepsilon_{n}-\varepsilon_{n+2}\right) \ldots\left(\varepsilon_{n}-\varepsilon_{n+a_{i}}\right)
$$

For example:

$$
\begin{aligned}
n_{1} & =\left(-\alpha_{n}\right)\left(s_{n} s_{n+1} \ldots s_{n+a_{1}-1}\left(-\alpha_{n}\right)\right) \ldots\left(\left(s_{n+1} \ldots s_{n+a_{1}-1}\right) \ldots\left(s_{n} s_{n+1}\right)\left(-\alpha_{n}\right)\right) \\
& =\left(\varepsilon_{n}-\varepsilon_{n+1}\right)\left(\varepsilon_{n}-\varepsilon_{n+2}\right) \ldots\left(\varepsilon_{n}-\varepsilon_{n+a_{1}}\right)
\end{aligned}
$$

The lemma now follows.

Remark 9.2. One might hope that the reason that the Euler class of the normal bundle to $F \subset f^{-1}\left(S_{w_{A}}\right)$ is combinatorial is because it is an induced bundle as in Example 8.1. I was unable to decide whether this is the case.

Proof of Theorem 1.3. We can now complete the proof of theorem 1.3. If we denote by $c \in H_{T}^{*}(F)$ the Euler class of the normal bundle to $F \subset f^{-1}\left(w_{A} P_{M} / P_{M}\right)$ and $p: F \rightarrow$ pt denotes the projection then, by Lemma 9.1 and Corollary 8.4 we have

$$
p_{!}(c)=\partial_{w_{m}}\left(n_{m} \partial_{w_{m-1}}\left(n_{m-1} \ldots\left(\partial_{w_{1}} n_{1}\right) \ldots\right)\right)
$$

By repeated application of Lemma 9.3 below
$\partial_{w_{m}}\left(n_{m} \partial_{w_{m-1}}\left(n_{m-1} \ldots\left(\partial_{w_{1}} n_{1}\right) \ldots\right)\right)=\partial_{w_{m}}\left(\varepsilon_{n}^{a_{m}} \partial_{w_{m-1}}\left(\varepsilon_{n}^{a_{m-1}} \ldots\left(\partial_{w_{1}} \varepsilon_{n}^{a_{1}}\right) \ldots\right)\right)=C$
where $C$ is as in the introduction. This completes the proof.
Lemma 9.3. Consider an expression of the form

$$
D=\partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}+h_{m}^{+} g_{m}^{\prime}\right) \ldots\right)\right) \in R
$$

with $u_{i} \in W_{M}, g_{i} \in R, g_{m}^{\prime} \in R$ and $h_{m}^{+} \in R^{W_{M}}$. If $D \in \mathbb{Z}$ and $h_{m}^{+}$is of degree $>0$ then

$$
D=\partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}\right) \ldots\right)\right)
$$

Proof. Because $\partial_{u_{i}}\left(h_{m}^{+} g\right)=h_{m}^{+} \partial_{u_{i}}(g)$ for all $g \in R$ we have

$$
\begin{gathered}
D=\partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}+h_{m}^{+} g_{m}^{\prime}\right) \ldots\right)\right)= \\
=\partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}\right) \ldots\right)\right)+h_{m}^{+} \partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}^{\prime}\right) \ldots\right)\right) .
\end{gathered}
$$

Because $D \in \mathbb{Z}$, the term $\partial_{u_{1}}\left(g_{1} \partial_{u_{2}}\left(g_{2} \ldots \partial_{u_{m}}\left(g_{m}^{\prime}\right) \ldots\right)\right)$ is of negative degree, and hence is zero. The lemma follows.

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[^0]:    ${ }^{1}$ The link of a point $x \in X$ is defined as follows: first we embed an affine neighbourhood of $x$ into $\mathbb{C}^{N}$ so that $x \mapsto 0$; then the link is defined to be $X \cap S_{\varepsilon}^{2 N-1}$ for small $\varepsilon$, where $S_{\varepsilon}^{2 N-1} \subset \mathbb{C}^{N}$ denotes the sphere of radius $\varepsilon$ centred at the origin.

[^1]:    ${ }^{2}$ In this paper resolution of singularities is used to refer to any proper birational morphism of algebraic varieties with smooth source. We do not require our map to be an isomorphism over the smooth locus of $X$.

